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PROGRESS REPORT.

It is with great sorrow that the Committee announce the sad death, in England on 19th July 1915, of their President, Principal E. W. Middlemast of the Presidency College, Madras, who has been taken away from their midst all too soon. Under his fostering and sympathetic guidance the Society looked forward to a career of greater usefulness and prosperity. In his death, Madras has lost a professor of great note, the mathematical world a scholar of distinction, and the society a tried and trusted guide.

2. The following gentleman have been elected as members at concessional rate—

1. *Mr. K. S. Srinivasachariar, B.A., L.T.*—Senior Mathematics Assistant, Multhialpet High School, No. 179, Thumbu Chetty Street, Madras;
2. *Mr. Pragada Satyanarayana, B.A.*—Mathematics Assistant, C. B. M. School, Vizagapatam;
3. *Mr. M. Vaidyanathan, M.A., L.T.*—No. 22, Vellala Street, Purasawakam, Madras.

3. The following books have been received—

1. Bombay University Calendar for 1915, Vols. I & II.
2. (*a*) Practical Geometry for Indian Schools, and (*b*) Practical Geometry for Indian Colleges—by R. D. Karve, Geo. Bell & Sons, London, 1915.

4. Facultad De Ciencias Fisicas, Mathematicas *y* Astronomicas of La Plata have arranged to send their publication in exchange for our Journal.

POONA,
30th Sept., 1915.

D. D. KAPADIA,
Hon. Joint Secretary.

Note on Discontinuous flow of fluid in a Diverging Canal.

By J. M. Bose, M.A., B.Sc.

The problem of discontinuous flow of liquid in two dimensions was first successfully solved by Helmholtz (*Berlin Monatsberichte*, April 1868) and his solution of some simple cases will be found in well-known text-books on Hydrodynamics (*e.g.* Lamb, pp. 85–101) In the *Philosophical Transactions* of the Royal Soc. 1890, Michell discussed a number of cases, and all those problems which have been solved so far will be found in a memoir by Sir G. Greenhill (Advisory Committee's Report, No. 19).

One of Michell's examples in which a direct solution is given is the case of discontinuous flow of liquid in a canal of uniform section, *i.e.* with parallel sides. In the present note I propose to give an extension of this problem, *i.e.* to the discontinuous flow of fluid in a diverging canal, like a river flowing into the sea. If the river banks terminate abruptly into an ocean of infinite expanse, then the stream lines which coincide with the banks become discontinuous at this point, leaving a region of dead water behind, which plays an important part in coast erosion.

The analytical method is an application of the Schwarz-Christoffel theory of conformal representation—a detailed treatment of which will be found in text-books on the Theory of Functions and an excellent chapter on its application to Hydrodynamics will be found in Ramsey's *Hydrodynamics*.

Our problem is to determine w or $\phi+i\psi$ as a function of z or $x+iy$, which satisfies certain boundary conditions. This is done by conformally transforming the curves in the w—plane and z—plane, into the upper half plane of an intermediate variable t. Schwarz-Christoffel's theorem then gives the required relation between w and z.

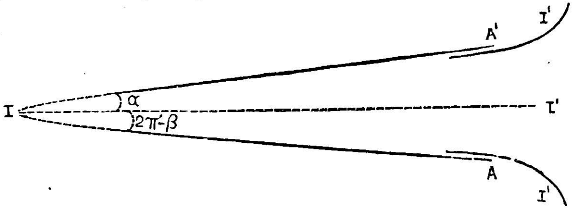
It is convenient however to use another notation Ω defined by

$$\Omega=\log\zeta=\log Q/q+i\Theta, \text{ and } \zeta=-Q\frac{dz}{dw}$$

where q and Θ specify the magnitude and direction of the resultant velocity at any point.

Along plane boundaries Θ is constant, and if there are no external forces then on the free surface q is constant and equal to Q, since the

pressure is constant (Bernoulli's theorem). Thus the figure in the Ω-plane is bounded by straight lines.



Let the above figure represent a section of the canal (*i.e.* the z-plane) and for w, Ω and t-diagrams, we refer to Lamb's *Hydrodynamics*, p. 90.

Here, in the w-diagram, AB corresponds to $\psi=m$ and A′ B′ to $\psi=-m'$ and in the Ω-diagram AB corresponds to $\theta=2\pi-\beta$ and A′ B′ to $\theta=\alpha$. Let the points A, A′, I in the t-plane correspond to $t=a, a', 0$ respectively.

The $w-t$ and $\Omega-t$ relations are given by

$$\frac{dw}{dt}=\frac{N}{t}, \text{ or } w=N\log t+P;$$

$$\frac{d\Omega}{dt}=\frac{M}{t\sqrt{(t-a_1)(t-a')}}, \text{ or } \Omega=\frac{-M}{\sqrt{aa'}}.\ 2\cosh^{-1}\sqrt{\frac{a(t-a')}{(a-a')t}}+Q;$$

where $\infty>a>a'>-\infty$ in the t-plane, and P, Q, M, N are constants to be determined.

We have, when $t=a',\ \Omega=(2\pi-\beta)i$;
$t=a,\ \Omega=\alpha i.$

From these we get

$$(2\pi-\beta)i=\frac{-M}{\sqrt{aa'}}.\ 2\cosh^{-1}(0)+Q, \qquad \ldots \qquad \ldots (1)$$

$$\alpha i=\frac{-M}{\sqrt{aa'}}.\ 2\cosh^{-1}(1)+Q; \qquad \ldots \qquad \ldots (2)$$

whence
$$\frac{-M}{\sqrt{aa'}}=\frac{2\pi-\beta-\alpha}{\pi}$$

and
$$Q=\alpha i,$$

so that
$$\Omega=\frac{2\pi-(\alpha+\beta)}{\pi}.\ 2\cosh^{-1}\sqrt{\frac{a(t-a')}{(a-a')t}}+\alpha i;$$

along A′I′, $Q=q$ $\therefore$ $\Omega=\theta i$

$$\therefore \quad \cosh\left[\frac{\pi(\Omega-\alpha i)}{2(2\pi-\alpha+\beta)}\right]=\sqrt{\frac{a(t-a')}{(a-a')t}}$$

or
$$\cos^2\left[\frac{\pi}{2}\cdot\frac{\theta-\alpha}{2\pi-\alpha+\beta}\right]=\frac{a}{a-a'}\cdot\frac{t-a'}{t}, \qquad \ldots \qquad \ldots (3)$$

and $$\sin^2\left[\frac{\pi}{2}\cdot\frac{\theta-\alpha}{2\pi-\alpha+\beta}\right]=\frac{a'}{a-a'}\cdot\frac{a-t}{t}=\frac{a'}{a-a'}\left[\frac{a}{t}-1\right]. \quad \dots \quad (4)$$

Now remembering that w vanishes for $t=a$ and increases suddenly by $(m+m')i$ when t passes through the origin (*i.e.* changes sign), we have

$$w=\frac{m+m'}{\pi}\log\frac{a}{t},$$

also $$\sinh n(\Omega-\alpha i)=\sqrt{\frac{a'}{a-a'}\cdot\frac{t-a}{t}}$$

where $n=\frac{\pi}{2(2\pi-\alpha+\beta)}$.

Along the stream line A′I′

$$Q=q \text{ and } Qds=-d\phi=-dw;$$

$$Qs=\frac{m+m'}{\pi}\log\frac{t}{a}, \text{ or } t=ae^{\frac{\pi s}{c}},\ m+m'=Qc.$$

Also from (4)

$$\sin^2 n\,(\theta-\alpha)=\frac{a}{a-a'}\left(\frac{a}{t}-1\right)=\frac{a'}{a-a'}\left(e^{-\frac{\pi s}{c}}-1\right)$$

giving the intrinsic equation of the free stream line.

If β be the inclination of the free stream line at a sufficient distance, *i.e.* if $\theta=\beta$ when $t=\infty$,

$$\sin^2 n(\beta-\alpha)=-\frac{a'}{a-a'}.$$

Along the bank IA′

$$\theta=\alpha$$

$$\sinh\left(n\log\frac{Q}{q}\right)=\sqrt{\frac{a'}{a-a'}\cdot\frac{t-a}{t}}=\sqrt{T} \text{ say.}$$

Thus the velocity along the bank is given by

$$\left(\frac{Q}{q}\right)^n=\sqrt{T}+\sqrt{T+1};$$

or $$\frac{q}{Q}=\left[\cos n\,(\beta-\alpha)\sqrt{\frac{t-a'}{t}}-\sin n\,(\beta-\alpha)\sqrt{\frac{a-t}{t}}\right]^{\frac{1}{n}}.$$

If we put $\alpha=\beta$ and change the sign of w, we get the solution of Helmholtz's problem of Borda's re-entrant mouthpiece.

If there are a series of bends outwards, then the above method cannot be applied owing to the fact that the velocity becomes infinite at each bend. But the above method can be extended if the bends are inwards. The problem of flow along a lamina with a series of inward bends has been discussed in a paper by Bryan and Jones in the *Proc. of the Royal Soc.* (vol. 91, No. A 630.)

Squaring the Circle.*

By S. Krishnaswami Aiyangar, B.A.

The problem of "squaring the circle" has attracted the attention of the greatest of mathematicians from the earliest times up to the present day. It is a problem whose origin is to be found in the remote antiquity and by following its history one sees how "it proceeded and changed from age to age according to the progressive development of the mathematical science." The problem, as understood by the ancient Greeks, was the determination of a square whose area is equal to that of a circle of given radius, the corners of the square being determined by constructions involving straight lines and circles only; that is, by Euclidean geometry. But the only instruments permissible in Euclidean geometry are the compasses and the ungraduated straight ruler. Hence it is that the ancient Greeks attacked the problem armed only with these weapons.

Closely allied to this problem is that of the rectification of the circle. It was known at an early date that the ratio of the circumference of a circle to its diameter is constant in value and is the same as the ratio of the area of a circle to the square on its radius. From the time of the great French mathematician Euler this constant number has been represented by the Greek letter π.

If π represented merely the ratio of the circumference of a circle to its diameter, the determination of its value would lose much of its interest. "But it really represents a certain number which would enter into analysis from whatever side the subject is approached. To illustrate how little the usual definition of π suggests its manifold properties there is an anecdote. De Morgan of whom we have heard so much, was once explaining to an actuary 'what was the chance that a certain proportion of some group of people would be alive at the end of a given time' and he quoted the actuarial formula involving π; and in answer to a query, he said π represented the ratio of the circumference of a circle to its diamerter. His friend who was until then listening to him with great interest interrupted him and exclaimed 'my dear friend, that must be a delusion; what can a circle have to do with the number of people alive at the end of a given time.'

The history of our problem embraces three distinct periods. The first period begins with the empirical determination of the ratio of

* From a paper read before the Mathematics Association, Teachers' College, Saidapet, in 1914.

the circumference to its diameter and ends in the middle of the seventeenth century when the calculus was invented. This period was mainly geometrical and mathematicians tried to determine π by calculating the sides or areas of regular polygons inscribed or circumscribed to a circle and considering that the circumference or area of the circle always lay between these values. The closeness of the approximation obtained during this period, when arithmetical notation had not reached any degree of perfection, is truly surprising. At the end of this period the geometrical method had been perfected to such a high degree that no further progress could be hoped for on the lines laid down by the Greek mathematicians.

The second period commenced in the middle of the seventeenth century and lasted for about a hundred years. During this period the powerful analytical methods provided by the help of the calculus were successfully employed and we find an abundance of convergent series products and continued fractions to represent π, by means of which the value of the number was determined to a larger number of decimal places than was possible before. But the real nature of the number π was still unknown and the mathematicians of this period could not say whether the number was rational or irrational.

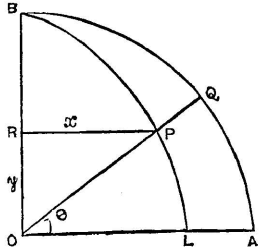
In the third period which lasted from the middle of the eighteenth century to the close of the nineteenth century the number was put to a very critical examination, and at last the discovery was made that the number did not belong to the category of 'algebraic numbers'. That is, it could not be a root of an algebraic equation with rational coefficients; and that it belonged to a class of numbers commonly known as transcendental. Thus it was conclusively proved that the number π could not be constructed by the use of Euclidean instruments or even by the use of higher algebraic curves besides the straight line and the circle.

Having thus briefly set forth the chief characteristics of the different periods, into which the history of the determination of π naturally falls, let us next examine at some length how the problem progressed according to the improvement of the general mathematical science in each period.

The earliest traces of the evaluation of π are to be found in the writings of the ancient Egyptians who used the formula: Area of a circle $=\left(\frac{8}{9}\right)^2 d^2$, which gives 3·1604...for the value of π. They also used the less accurate formula $\pi=3$.

After the ancient Egyptians came the early Greek mathematicians, who were the first to deal systematically with the problems of the rectification and quadrature of the circle. About the latter half of the fourth century B.C. Dinostratus invented and closely studied a curve called the Quadratrix by means of which he was able to find an approximate construction for π. This curve may be described as follows :—

AOB is a circular quadrant. Two points Q, R starting from A, O respectively travel along the arc AB and the radius OB and reach B at the same time. If from R a perpendicular is drawn to meet OQ at P then the locus of P will be the quadratrix. From the figure $y \propto \theta$ and $y=a(=OB)$ when $\theta=\frac{\pi}{2}$.



$$\therefore \qquad y=\frac{2a}{\pi}\theta.$$

From $\triangle$OPR, $x=y\cot\theta=y\cot\frac{\pi y}{2a}$.

$$\therefore \qquad OL=\underset{y=0}{Lt}\; y\cot\frac{\pi y}{2a}=\frac{2a}{\pi}.$$

Here the curve cuts the x—axis at a distance represented by $\frac{2a}{\pi}$ and hence an approximate geometrical construction of π is effected. But the construction of the curve involves the same difficulty as that of π. But a rigorous mathematical treatment of the subject begins with Archimedes who proved that π is less than $3\frac{1}{7}$ but greater than $3\frac{70}{71}$. He was able to effect this by inscribing in a circle and circumscribing about it regular polygons of 96 sides and assuming that the circumference of the circle must lie between the perimeters of these polygons. The essential point of this method translated into our modern notation

becomes $\sin\theta < \theta < \tan\theta$, when $\theta = \frac{\pi}{96}$. This process is closely allied to our modern method of limits and it is in this that its chief interest* lies. "But the immature condition of arithmetic at the time was the real obstacle preventing the evaluation of the ratio to any degree of accuracy whatever."

Before going to the European mathematicians of the fifteenth and sixteenth centuries let us see how the problem was developed in India and other eastern countries. Aryabhatta, the earliest of the Indian mathematicians, used the relation $\pi = \frac{62832}{20000}$, which is equivalent to 3·1416. He proved that if a_n be the side of a regular polygon of n sides inscribed in a circle of unit diameter, and Q_n that of a regular inscribed polygon of $2n$ sides then $Q_{2n} = \frac{1}{2} - \frac{1}{2}(1 - a_n^2)^{\frac{1}{2}}$. From the side of an inscribed hexagon he calculated the sides of polygons of 12, 24, 48, 96, 192 and 384 sides, the last result of which becomes $\sqrt{9{\cdot}8694}$, which gives the approximation quoted above. Of the other Hindu mathematicians, Brahmagupta used the approximation $\sqrt{10}$ and Bhaskaracharya 3·1416 Approximate values were used also by the Arabs and the Chinese. But the Japanese approximations were closer than the Arabs'.

Returning now to the European mathematicians, we find that after the time of the Renaissance Adriaen Anthonisz calculated that $\pi > \frac{333}{106}$ and $< \frac{377}{120}$. He boldly asserted that its true value is equal to the mean of the numerators divided by the mean of the denominators; that is, $\frac{355}{113}$, which is a very close approximation for all practical purposes. After him came Vieta, who, by a consideration of the areas of regular polygons of n sides and $2n$ sides inscribed in a circle of unit diameter, got an infinite sequence for the value of π. He proved that

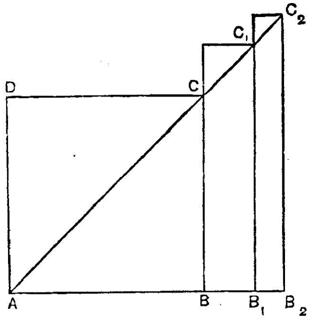
$$\frac{2}{\pi} = \sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\cdots\cdots$$

One writer remarks that "from this period onwards, therefore, no knowledge whatever of geometry was necessary, to any one who aspired to determine the ratio to any required degree of accuracy; the mere arithmetician's art and length of days were the only requisites."

Thus a genus of workers became possible who may be styled 'π-computers', a name which if it connotes anything uncomplimentary does so because of the almost entirely fruitless character of the labours."

* "Squaring the Circle" by E. W. Hobson, (1913) p. 31.

In connection with the development of the method of Archimedes we have to introduce two more names of importance: Huyghens and Gregory. Gregory attempted a proof of the impossibility of the quadrature by Archimedian methods, and Huyghens reputed Gregory's proof. "Huyghens expressed his own conviction of the impossibility of the quadrature, and in his controversy with Wallis remarked that it was not even decided whether the area of a circle and the square of the diameter are commensurable or not."



But Descartes the inventor of analytical geometry approached the problem from quite a new standpoint. He started with a given straight line equal to the circumference and tried to find its diameter by the following method. On AB $=\frac{1}{4}$ of the circumference a square ABCD is constructed; on AC produced a point C_1 is taken such that rect. $BC_1=\frac{1}{4}$ sq. ABCD; similarly C_2 is taken such that rect. $B_1C_2=\frac{1}{4}$ rect. BC_1; and so on, *ad infinitum*. Then the diameter is given by Lt. AB_n. By the construction it is easily seen that $AB_n.(AB_n-AB_{n-1})=\frac{1}{4}AB$, which is satisfied by $AB_n=\frac{4AB}{2^n}\cot\frac{\pi}{2^n}$.

$$\therefore \quad \underset{n\to\infty}{\text{Lt.}} AB_n=\frac{4AB}{} \underset{n\to\infty}{\text{Lt.}} \cos\frac{\pi}{2^n}\left[\frac{\pi}{2^n}\Big/\sin\frac{\pi}{2^n}\right]=\frac{4AB}{\pi}$$

The invention of logarithms by John Napier early in the seventeenth century greatly enhanced the mathematician's attention to the number π; thence forward we find the two numbers e and π going hand in hand until at last in the 19th century these two numbers received one of the greatest developments that modern mathematics can boast of. To this we shall return later.

As has been already pointed out, the invention of the calculus in the second half of the same century led to a radical change in the

ideas and methods of the mathematicians. After this date the geometrical phase of the problem was forgotten and mathematicians began to discover analytical expressions to represent π. This new method is first seen in the works of John Wallis who by an "ingenious method depending upon the expression for $\frac{\pi}{8}$ the area of a semi-circle of unit diameter as the definite integral $\int_0^1 (x-x^2)^{\frac{1}{2}}dx$" invented his well-known expression

$$\frac{\pi}{2}=\frac{2\cdot4\cdot6\cdot\ldots}{1\cdot3\cdot5\cdot\ldots}$$

But the expression most frequently used for calculating π is that given by Gregory: $\tan^{-1}x=x-\frac{x^3}{3}+\frac{x^5}{5}+\ldots$ This was afterwards independently discovered by the French mathematician Leibnitz and the particular case got by substituting $x=1$ is known as Leibnitz's series. By means of Gregory's series Abraham Sharp was able to calculate the value of π to 72 decimal places. By taking the more convergent series for $\sin^{-1}x$ Newton calculated π for 14 places. In 1737 Euler employed the equation

$$\frac{\pi}{4}=\tan^{-1}\tfrac{1}{2}+\tan^{-1}\frac{1}{2\cdot4}+\tan^{-1}\frac{1}{2\cdot9}.$$

In 1746 a still more convergent series was used by Machin from which π was calculated for 100 places. "More astonishing still have been the deeds of the π computers of the 19th century." A condensed record compiled by Glaisher is given below for those interested in the subject.

Date.	Computor.	No. of digits calculated.	No. of digits correct.
1842	Rutherfield	208	152
1844	Dase	205	200
1847	Clausen	250	248
1853	Shankes	318	318
1853	Rutherfield	440	440
1853	Shanks	530	......
1853	Shanks	607	......
1853	Richter	333	330
1854	Richter	400	330
1854	Richter	400	400
1854	Richter	500	500
1873	Shanks	707	......

In this connection it may be remarked that for most mathematical calculations the value of π to 6 or 7 decimal places is quite sufficient. It has been calculated that if a circle is drawn on a distance of 200 miles as diameter and its actual circumference measured, then its value calculated by taking π to 15 decimal places will differ from the actual value by less than 18 millionth part of a millimetre. Such being the case with 15 decimal places, it need hardly be pointed out the impossibility of the human mind to conceive the degree of accuracy obtained by following the value of π for 500 or for 700 decimal places.

Before going to the third and the final period of the history of our problem let us see how its development underwent changes in the hands of the French mathematician Euler. It was he who first introduced the idea of ratio in place of the old sine, cosine, tangent etc. which were simply the lengths of straight lines connected with the circular arc. "Thus trigonometrical ratios became functions of an angular magnitude and therefore numbers." This mode of regarding trigonometrical ratios as analytical functions gave rise to one of the grandest discoveries in modern mathematics. Representing $1+x+\frac{x^2}{2!}+\frac{^3}{3!}+\ldots$ by e^x he proved that $\cos = \frac{e^{xi}+e^{-xi}}{2}$ and $\sin x = \frac{e^{xi}-e^{-xi}}{2i}$ which can be written in the form $e^{xi} = \cos x + i\sin x$ and $e^{-xi} = \cos x - i\sin x$. Putting $x=\pi$ in the first relation, we get $e^{\pi i} = -1$, a fundamental relation between the two numbers e and π which is of the greatest importance in proving the real nature of π. Though much improvement was made in this period nothing was yet known as to the real nature of π. In this connection it is interesting to hear what Legendre the French mathematician has to say. In his "Elements of geometry" published in the year 1794 he writes: "It is probable that the number π is not even contained among the algebraical irrationalities, that is, it cannot be a root of an algebraic equation, with a finite number of terms whose co-efficients are rational. But it seems to be very difficult to prove this strictly."

It was in the third and final period that our problem reached its highest point of development. Lambert, the German mathematician who flourished during the years 1728—77 expressed tan z in the form of an infinite continued fraction and hence proved that if x is a rational number different from zero, then tan x cannot be a rational number. If $x=\frac{\pi}{4}$ we have $\tan x = 1$ and therefore $\frac{\pi}{4}$ or π cannot be a rational number.

In 1882 Lindemann, a German mathematician proved that if x_1, x_2, x_3,.........are any real or complex algebraical numbers, all distinct and p_1, p_2, p_3,.........are n algebraical numbers at least one of which is different from zero, then the sum $p_1 e^{x_1} + p_2 e^{x_2} + p_3 e^{x_3} + \ldots + p_n e^{x_n} \neq 0$. Therefore as a particular case $e^{xi} + 1 \neq 0$. But we know that $e^{\pi i} + 1 = 0$. Therefore π is not algebraic but transcendental.

Now, we know that every problem in pure geometry has its corresponding counter-part in analytical geometry and it can be easily proved that the determination of a geometrical construction by means of Euclidean methods corresponds, in co-ordinate geometry, to the solution of "only those algebraic equations which are obtainable by elimination from a sequence of linear and quadratic equations." But "π being transcendental, is not a root of any algebraic equation at all;" and therefore in accordance with the criterion mentioned, cannot be determined by Euclidean construction.

Thus it has been conclusively proved once for all that the "ideal of an exact construction" by means of Euclidean methods is an impossibility. In the words of Schubert "these are the words of a final determination of a controversy which is as old as the history of the human mind. But the race of circle squarers unmindful of the verdict of mathematics, the most infallible of arbiters, will never die out as long as ignorance and thirst for glory remain united."

But if an actual construction of π is desired it can only be effected by means of a "transcendental" curve. This necessitates the use of the transcendental apparatus "Integraph" which was recently invented and described by a Russian engineer. If the linkwork of the Integraph is moved in such a manner that the guiding point describes the curve $y=f(x)$, then the tracing point will trace out the integral curve $y=\int f(x)dx$. Taking for the differential curve the circle $x^2+y^2=a^2$, we see that the integral curve is

$$y=\int \sqrt{(a^2-x^2)}dx$$

$$=\frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a},$$

which cuts the y-axis in points whose ordinates are o, $\pm a^2\frac{\pi}{2}$, etc.

SHORT NOTES.

On the sum of the square roots of the first n Natural Numbers.

1. Let

$$\phi_1(n)=\sqrt{1}+\sqrt{2}+\sqrt{3}+\ldots+\sqrt{n}-(C_1+\tfrac{2}{3}n\sqrt{n}+\tfrac{1}{2}\sqrt{n})$$
$$-\frac{1}{6}\sum_{\nu=0}^{\nu=\infty}(\sqrt{n+\nu}+\sqrt{n+\nu+1})^{-3}$$

where C_1 is a constant such that $\phi_1(1)=0$. Then we see that

$$\phi_1(n)-\phi_1(n+1)=-\sqrt{n+1}+[\tfrac{2}{3}(n+1)\sqrt{n+1}+\tfrac{1}{2}\sqrt{n+1}]$$
$$-(\tfrac{2}{3}n\sqrt{n}+\tfrac{1}{2}\sqrt{n})+\tfrac{1}{6}(\sqrt{n}-\sqrt{n+1})^3=0.$$

But $\phi_1(1)=0$. Hence $\phi_1(n)=0$ for all values of n. That is to say

$$(1)\quad \sqrt{1}+\sqrt{2}+\sqrt{3}+\sqrt{4}+\ldots+\sqrt{n}=C_1+\tfrac{2}{3}n\sqrt{n}+\tfrac{1}{2}\sqrt{n}$$
$$+\tfrac{1}{6}\{(\sqrt{n}+\sqrt{n+1})^{-3}+(\sqrt{n+1}+\sqrt{n+2})^{-3}$$
$$+(\sqrt{n+2}+\sqrt{n+3})^{-3}+\ldots\}.$$

But it is known that

$$(2)\quad C_1=-\frac{1}{4\pi}\left(\frac{1}{1\sqrt{1}}+\frac{1}{2\sqrt{2}}+\frac{1}{3\sqrt{3}}+\ldots\ldots\right)$$

Putting $n=1$ in (1) and using (2), we obtain

$$(3)\quad 2\pi\left\{\frac{1}{(\sqrt{1})^3}+\frac{1}{(\sqrt{1}+\sqrt{2})^3}+\frac{1}{(\sqrt{2}+\sqrt{3})^3}+\frac{1}{(\sqrt{3}+\sqrt{4})^3}+\ldots\right\}$$
$$=3\left\{\frac{1}{(\sqrt{1})^3}+\frac{1}{(\sqrt{2})^3}+\frac{1}{(\sqrt{3})^3}+\frac{1}{(\sqrt{4})^3}+\ldots\right\}$$

2. Again let

$$\phi_2(n)=1\sqrt{1}+2\sqrt{2}\ldots+n\sqrt{n}-(C_2+\tfrac{2}{5}n^2\sqrt{n}+\tfrac{1}{2}n\sqrt{n}+\tfrac{1}{8}\sqrt{n})$$
$$-\frac{1}{40}\sum_{\nu=0}^{\nu=\infty}[\sqrt{(n+\nu)}+\sqrt{(n+\nu+1)}]^{-5}$$

where C_2 is a constant such that $\phi_2(1)=0$. Then we have

$$\phi_2(n)-\phi_2(n+1)=-(n+1)\sqrt{n+1}+$$
$$\{\tfrac{2}{5}(n+1)^2\sqrt{n+1}+\tfrac{1}{2}(n+1)\sqrt{n+1}+\tfrac{1}{8}\sqrt{n+1}\}$$
$$-\{\tfrac{2}{5}n^2\sqrt{n}+\tfrac{1}{2}n\sqrt{n}+\tfrac{1}{8}\sqrt{n}\}+\tfrac{1}{40}(\sqrt{n}-\sqrt{n+1})^5=0.$$

But $\phi_2(1)=0$. Hence $\phi_2(n)=0$. In other words

$$(4)\quad 1\sqrt{1}+2\sqrt{2}+3\sqrt{3}+\ldots n\sqrt{n}=C_2+\tfrac{2}{5}n^2\sqrt{n}+\tfrac{1}{2}n\sqrt{n}+\tfrac{1}{8}\sqrt{n}$$
$$+\frac{1}{40}\{(\sqrt{n}+\sqrt{n+1})^{-5}+(\sqrt{n+1}+\sqrt{n+2})^{-5}$$
$$+\sqrt{n+2}+\sqrt{n+3})^{-5}+\ldots\}.$$

But it is known that

$$(5)\quad C_2=-\frac{3}{16\pi^2}\left(\frac{1}{1^2\sqrt{1}}+\frac{1}{2^2\sqrt{2}}+\frac{1}{3^2\sqrt{3}}+\ldots\right).$$

It is easy to see from (4) and (5) that

$$(6)\quad 2\pi^2\left\{\frac{1}{(\sqrt{1})^5}+\frac{1}{(\sqrt{1}+\sqrt{2})^5}+\frac{1}{(\sqrt{2}+\sqrt{3})^5}+\frac{1}{(\sqrt{3}+\sqrt{4})5}+\dots\right\}$$
$$=15\left\{\frac{1}{(\sqrt{1})^5}+\frac{1}{(\sqrt{2})^5}+\frac{1}{(\sqrt{3})^5}+\frac{1}{(\sqrt{4})^5}+\dots\right\}$$

3. The corresponding results for higher powers are not so neat as the previous ones. Thus for example

$$(7)\quad 1^2\sqrt{1}+2^2\sqrt{2}+3^2\sqrt{3}+\dots+n^2\sqrt{n}=C_3+\sqrt{n}\left(\frac{2}{7}n^3+\tfrac{1}{2}n^2+\frac{5}{24}n\right)$$
$$-\frac{1}{96}\{(\sqrt{n}+\sqrt{n+1})^{-3}+(\sqrt{n+1}+\sqrt{n+2})^{-3}+\dots\}$$
$$+\frac{1}{224}\{(\sqrt{n}+\sqrt{n+1})^{-7}+(\sqrt{n+1}+\sqrt{n+2})^{-7}$$
$$+(\sqrt{n+2}+\sqrt{n+3})^{-7}+\dots\};$$

$$(8)\quad 1^3\sqrt{1}+2^3\sqrt{2}+\dots+n^3\sqrt{n}=C_4+\sqrt{n}\left(\frac{2}{9}n^4+\tfrac{1}{2}n^3+\frac{}{24}n^2-\frac{7}{384}\right)$$
$$-\frac{1}{192}\{(\sqrt{n}+\sqrt{n+1})^{-5}+(\sqrt{n+1}+\sqrt{n+2})^{-5}+\dots\}$$
$$+\frac{1}{1152}\{(\sqrt{n}\div\sqrt{n+1})^{-9}+(\sqrt{n+1}+\sqrt{n+2})^{-9}+\dots\};$$

and so on.

The constants C_3, C_4,...can be ascertained from the well known result, *viz. the constant in the approximate summation of the series* $1^{r-1}+2^{r-1}+3^{r-1}+\dots+n^{r-1}$ *is*

$$(9)\qquad \frac{2\Gamma(r)}{(2\pi)^r}\left(\frac{1}{1^r}+\frac{1}{2^r}+\frac{1}{3^r}+\frac{1}{4^r}+\dots\right)\cos\tfrac{1}{2}\pi r,$$

provided that the real part of r is greater than 1.

4. Similarly we can show, by induction, that

$$(10)\qquad \frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\dots\dots+\frac{1}{\sqrt{n}}=C_0+2\sqrt{n}+\frac{1}{2\sqrt{n}}$$
$$-\tfrac{1}{2}\left\{\frac{(\sqrt{n}+\sqrt{n+1})^{-3}}{\sqrt{n(n+1)}}+\frac{(\sqrt{n+1}+\sqrt{n+2})^{-3}}{\sqrt{(n+1)(n+2)}}+\dots\dots\right\}$$

The value of C_0 can be determined as follows: from (10) we have

$$(11)\quad \frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\dots\dots+\frac{1}{\sqrt{(2n)}}-2\sqrt{(2n)}\to C_0,$$

as $n\to\infty$. Also

$$(12)\quad 2\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{4}}+\frac{1}{\sqrt{6}}+\dots\dots+\frac{1}{\sqrt{(2n)}}\right)-2\sqrt{(2n)}\to C_0\sqrt{2}$$

as $n\to\infty$.

Now subtracting (12) from (11) we see that

$$\frac{1}{\sqrt{1}}-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\ldots\ldots-\frac{1}{\sqrt{(2n)}}\to C_0(1-\sqrt{2}),\text{ as } n\to\infty.$$

That is to say

$$\text{(13)}\quad C_0=-(1+\sqrt{2})\left(\frac{1}{\sqrt{1}}-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\ldots\ldots\right).$$

We can also show, by induction, that

$$\text{(14)}\quad \sqrt{1}+\sqrt{2}+\sqrt{3}+\ldots\ldots+\sqrt{n}=C_1+\frac{2}{3}n\sqrt{n}+\frac{1}{2}\sqrt{n}+\frac{1}{24\sqrt{n}}$$

$$-\frac{1}{24}\left\{\frac{(\sqrt{n}+\sqrt{n+1})^{-5}}{\sqrt{n(n+1)}}+\frac{(\sqrt{n+1}+\sqrt{n+2})^{-5}}{\sqrt{(n+1)(n+2)}}+\ldots\ldots\right\}$$

The asymptotic expansion of $(\sqrt{1}+\sqrt{2}+\sqrt{3}+\ldots+\sqrt{n})$ for large values of n can be shown to be

$$\text{(15)}\quad C_1+\frac{2}{3}n\sqrt{n}+\frac{1}{2}\sqrt{n}+\frac{1}{\sqrt{n}}\left(\frac{1}{24}-\frac{1}{1920n^2}+\frac{1}{9216n^4}-\ldots\ldots\right)$$

by using Euler-Maclaurin-sum Formula.

S. Ramanujan.

Definition of a Plane Curve.

Since it appears from Prof. Seshu Aiyar's note in a recent number of the Journal (April, 1915), that the problem of defining a plane curve is one in which readers of the Journal take interest, the following account of a definition hitherto unpublished may prove welcome. Limitations of time and space alike prevent me from giving examples illustrating either definitions previously given, such as are to be found in Young's treatise on "*The Theory of Sets of Points*", or the definition now offered; but any reader to whom the topic is not new will find the construction of examples easy.

A point O is called a limiting point of a set if there is no circle with O for centre and with radius different from zero which does not contain a point of the set other than O. This is one form of the classical definition, which we include to make this paper self-contained. It must be realised that O may or may not belong to a set of which it is a limiting point; whether O belongs to a set is entirely irrelevant to the question as to whether O is a limiting point of the set.

A set of points is called 'limited' if it does not extend to infinity, 'complete' if it contains all its limiting points, and 'closed' if it is both limited and complete. Strictly speaking, it is only in a region which is itself complete that limited sets are to be identified with sets that

do not extend to infinity, but for the purpose of defining a plane curve the identification may be made for the sake of brevity.

A set of points is said to be 'completed' if to it are added all of its limiting points which originally it does not include, and two sets are said to be 'inseparable' if the sets obtained by completing them have a common point. These definitions are applicable equally to limited sets and to sets which extend to infinity; their use with unlimited sets does not require the hypothesis of points at infinity, and a branch of a hyperbola, for example, is naturally said to be separable from each of the asymptotes.

A set of points is said to be 'connected' if in every partition of it into two sets the two components are inseparable. This definition may be compared with Jordan's definition of a 'connex' or connected set, accessible to readers of English in Young's treatise mentioned above (p. 204, Theorem 20) and in Hobson's treatise on "*The Theory of Functions of a Real Variable*" (pp. 137-138); as applied to sets which do not extend to infinity the definitions are equivalent; but a hyperbola and one of its asymptotes, for example, form a set connected in accordance with Cantor's difinition, with which Jordan compares his own, but not in accordance with the definition just given.

A set of points is said to be 'perfectly' connected if every pair of points belonging to it belongs to a closed connected set contained in the original set. This is the definition which is of fundamental importance; its uses extend far beyond the application to the definition of a curve. One example must suffice to shew its bearing: if a single point is removed from a circle, the set that remains is perfectly connected, though it is of course not complete, but if two points are removed, the incomplete set that remains is not perfectly connected but consists of two perfectly connected parts; the removal of any finite number of points or of an enumerably infinite number of points from the set composing a circle leaves a set which is connected in the sense of the definition above, as well as in Jordans sense and in Cantor's, but it is not a perfectly connected set if more points than one are taken.

The 'edge' of a set is the set composed of those points of the set which are limiting points of its complement. This is the common definition, which we repeat for the sake of readers to whom it is not familiar; the complement of a set, in the case of plane sets, is simply the set composed of all those points of the plane which do not belong to the set under consideration. To remove a trivial exceptional case we require one other elementary definition: *A 'multiple' set is a set containing more than one point.* We can now give the definition proposed for a plane curve.

A plane curve is a multiple perfectly connected plane set which coincides with its own edge. To say that a set coincides with its edge is one of the simplest ways of saying that every point of the set is a limiting point of the complement; the boundary of a set is defined as the sum of the edge of the set and the edge of the complement, and a set which coincides with its edge is a set which is contained in its boundary; a set is called *capacious* if it contains points which do not belong to the edge, and a set which coincides with its edge is a set which is nos capacious.

If a set is perfectly connected, so also is the set obtained by its completion, but a set which is not capacious may give on completion a capacious set. We can assert that if the set obtained by completing a plane curve is not capacious it is itself a plane curve, and it is found that an important classification of plane curves as we define them is into those which give capacious sets and those which give curves when they are completed.

The reader will realise that the definition of a curve given in this paper excludes the "space filling curves" of Peano and Moore. This exclusion is however no disadvantage, and is clearly inevitable in any definition of a purely descriptive type (see also Young's remarks on pp. 167-168 of the book already quoted.)

It is obvious that the method used to define a curve in a plane can be used to define a curve in any surface, the only modification necessary being in the definition of the complement of a set. But if the universe containig the sets is three-dimensional, surfaces as well as curves are multiple perfectly connected sets coinciding with their edges, and the manner in which a distinction can be made between curves and surfaces I can not attempt to describe in this paper; a full account of a series of definitions is already written, and notice of its publication shall be given in the Journal.

ERIC H. NEVILLE.

The Problem of Dirichlet.

(In ths case of a sphere.)

[This article is adapted from French mathematical books, viz., those of Poincare, Picard and Jordan and is given here for the benefit of those who cannot read French, in as much as Dirichlet's Problem is not adequately treated in any of the English books, at least not from a purist's stand point.]

Definition. A function V of the variables x, y, z which, with its partial derivatives of the first two orders, is continuous in a volume T bounded by a closed surface S and which satisfies in the volume T the equation of Laplace; *viz.*,

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=0$$

is called *a harmonic function* in the region T.

Enunciation of the Problem of Dirichlet. If we are given upon a closed surface S, a continuous succession of values, associated each with a point of the surface, does there always exist a harmonic function V continuous in the volume bounded by S and taking at the various points of S the system of values given? If so how to determine it?

This is the celebrated *interior* problem of Dirichlet which has been attacked by many eminent Mathematicians and which plays an important *rôle* in the Theory of Potentials.

As applied to 'Potentials', the problem may be enunciated thus:— "Given the Newtonian Potential upon a closed surface S which contains no attracting matter within it, find the potential at any point inside S."

There is a corresponding *exterior* problem of Dirichlet for the enunciation of which refer to any of the books named in the next paragraph.

That there is a solution to this problem has been proved by Riemann (for the proof, *vide*: Pierce's 'Newtonian Potential Function' page 104 or Thomson and Tait's 'Natural Philosophy', page 170); but this proof has been pointed out to be defective by Weierstrass (For Weierstrass's objections to Reimann's proof *vide*: Picard's '*Trait D'Analys*' vol. II, page 39.)

This problem has been solved by Neumann for any convex surface and also by Poincare (for these solutions *vide*: Poincare's '*Theorie du Potential Newtonian*); we shall however restrict ourselves in this article to the case of a sphere. Before proceeding to the actual solution we give below certain theorems which will be required in the solution.

2. (*a*) *Green's Theorem.* If U & V be two functions which, with their first derivatives are continuous in the volume T bounded by a closed surface S and which have second derivatives that are finite and integrable, then

$$(1)\quad \iint\left(U\frac{dV}{dn}-V\frac{dU}{dn}\right)d\sigma=\iiint\left(U\nabla^2V-V\nabla^2U\right)dxdydn.$$

the surface integral being extended to the bounding surface S and the triple integral being taken throughout the volume T. [For proof *vide* Williamson's *Integral Calculus* or Routh's *Analytical Statis*, Vol. II,]

The theorem is true whatever be the number of surfaces which bound the volume; only it should be noted that the derivatives $\frac{dU}{dn}$ and $\frac{dV}{dn}$ are taken along the normal exterior to the volume considered.

Cor. 1. If U & V are harmonic in T, $\nabla^2 U = 0 = \nabla^2 V$; then the Right side of (1) vanishes, and we have

$$(2) \qquad \iint \left(U\frac{dV}{dn} - V\frac{dU}{dn} \right) d\sigma = 0.$$

Cor. 2. In particular, if U=1 and V is harmonic, $\frac{dU}{dn}=0$ and (2) reduces to

$$(3) \qquad \iint \frac{dV}{dn} d\sigma = 0.$$

(*b*). If (a, b, c) be the co-ordinates of a fixed point A in the volume T bounded by S, and (x, y, z) the co-ordinates of a variable point P, then the function

$$\frac{1}{r} = \frac{1}{\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}}$$

satisfies the equation of Laplace. [*vide*: Routh or Williamson.]

It is not however continuous in T since it becomes infinite at the point A. But it is continuous in the region T_1 obtained by taking away from T a sphere Σ of infinitely small radius ρ and having A for its centre. Applying therefore (2) of *Cor.* 1 given above to $\frac{1}{r}$ and a harmonic function V in the region T_1, we have

$$\iint_S \left(V\frac{d\left(\frac{1}{r}\right)}{dn} - \frac{1}{r}\frac{dV}{dn} \right) d\sigma + \iint_\Sigma \left(V\frac{d\left(\frac{1}{r}\right)}{dn} - \frac{1}{r}\frac{dV}{dn} \right) d\sigma = 0,$$

the first integral being taken over the bounding surface S of the volume T and the second integral over the small sphere Σ described above, the differentiation $\frac{d}{dn}$ being along the normal interior to T_1.

It is easy to evaluate the second integral. For, upon the sphere Σ r has a constant value ρ, the second term of the integral is therefore equal to $-\frac{1}{\rho}\iint \frac{dV}{dn} d\sigma$ which is zero by *Cor.* 2.

$$\text{Also} \qquad \frac{d\left(\frac{1}{r}\right)}{dn} = -\frac{1}{r^2}\frac{dr}{dn} = -\frac{1}{\rho^2}$$

[for the direction of r is the direction of the normal exterior to the sphere, *i.e.* interior to the volume T_1; thus dr and dn are equal and have the same sign.]

$$\therefore \qquad \iint V \frac{d\left(\frac{1}{r}\right)}{dn} d\sigma = -\frac{1}{\rho^2}\iint V\, d\sigma.$$

Further the sphere Σ being infinitely small, the value of V at each of the points on it differs infinitely little from the value of V at the point A. This integral will therefore have for its limiting value

$$-V(a,b,c).\frac{1}{\rho^2}\iint d\sigma = -V(a,b,c,)4\pi\ ;$$

we have therefore

$$(4) \qquad 4\pi V(a,b,c) = \iint_S \left(V \frac{d\left(\frac{1}{r}\right)}{dn} - \frac{1}{r}\frac{dV}{dn} \right) d\sigma$$

$\frac{d}{dn}$ being taken along the normal interior to T.

(c) Suppose in particular that T is a sphere of radius R with (a, b, c) for its centre; then the formula (4) becomes, as shown above,

$$4\pi V(a,b,c) = \iint V \frac{d\left(\frac{1}{r}\right)}{dn} d\sigma,$$

$\frac{d}{dn}$ being along the normal interior to T, $\frac{d\left(\frac{1}{r}\right)}{dn}$ will, in this case, be $\frac{1}{R^2}$.

$$(5) \quad \therefore \quad 4\pi V\ (a,b,c) = \frac{1}{R^2}\iint V d\sigma.$$

(d) A harmonic function V continuous in a volume T cannot have either a maximum or a minimum value in the interior of the volume T. If possible let V be a maximum at the point (a, b, c) in the volume T.; describe a sphere Σ of very small radius ρ about (a, b, c) as centre; then at every point on this sphere

$$V(x, y, z.) < V(a, b, c)$$

$$\therefore \qquad \iint_\Sigma \frac{V d\sigma}{4\pi\rho} < V\ (a, b, c) \iint_\Sigma \frac{d\sigma}{4\pi\rho^2}$$

$$\therefore \qquad \frac{1}{4\pi\rho^2}\iint_\Sigma V d\sigma < V(a, b, c)$$

which is contrary to result (5) given above. Hence V cannot be a maximum; similarly it cannot be a minimum.

3. Having established these preliminary theorems we proceed to the problem of Dirichlet. Firstly we shall show that if the problem admits of a solution then it has only that one solution.

If possible let there be two solutions U and V to the problem; put $U - V = W$.

Now since U is a solution, $\nabla^2 U=0$ in the volume considered; similarly $\nabla^2 V=0$ in the same volume; also by our assumption $U=V$ on the surface S, both taking upon S the given system of values.

$\therefore$ $\nabla^2 W=0$ in the volume T and $W=0$ on the surface S.

$\therefore$ W must be identically zero in T; otherwise, it must have either a maximum or a minimum value in T which is contrary to the theorem proved in (*d*).

Hence $U\equiv V$ in the volume T. There cannot therefore be two distinct solutions of the problem of Dirichlet.

4. Thus far we have been talking of *any* closed surface S bounding a volume T. Now we pass on to the particular case of a sphere.

Assuming provisionally that there is a solution of the problem in the case of a sphere, we shall determine its form and then we shall show that the function we have obtained satisfies all the conditions of the problem.

Let R be the radius of the sphere and let (a, b, c) be the co-ordinates of a point A in its interior referred to the centre as origin and let l $(=\sqrt{a^2+b^2+c^2})$ be its distance from the centre; then we have, according to the formula (4) of § 2,

$$4\pi V(a, b, c)=\iint\left(V\frac{d\left(\frac{1}{r}\right)}{dn}-\frac{1}{r}\frac{dV}{dn}\right)d\sigma.$$

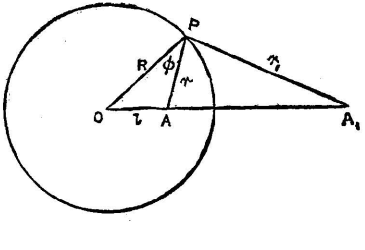
In the integral on the right hand side occurs besides V which is given on the surface, the derivative $\frac{dV}{dn}$ which is not given in the question. We use the following artifice to eliminate it:

Let A_1 be the point conjugate to the point A with respect to the sphere; then its distance l_1 from the centre is given by the formula

$$ll_1=R^2,$$

and if r, r_1 be the distances of any point on the sphere from the points A and A_1, we have

$$\frac{r}{r_1}=\frac{l}{R}$$



Now applying Green's theorem to the function V and to the function $\frac{1}{r_1}$ which is harmonic in the sphere, we have

$$\iint\left(V\frac{d\left(\frac{1}{r_1}\right)}{dn}-\frac{1}{r_1}\frac{dV}{dn}\right)d\sigma=0.$$

Multiply this equation by $\frac{R}{l}$ and subtract it from the similar one given above; then we get

$$4\pi V(a, b, c)=\iint V\left(\frac{d\left(\frac{1}{r}\right)}{dn}-\frac{R}{l}\frac{d\left(\frac{1}{r_1}\right)}{dn}\right)d\sigma;$$

but we have
$$\frac{d\left(\frac{1}{r}\right)}{dn}=-\frac{1}{r^2}\frac{dr}{dn}$$
$$=\frac{\cos\phi}{r^2}$$
$$=\frac{1}{r^2}\frac{R^2+r^2-l^2}{2Rr}.$$

Similarly
$$\frac{d\left(\frac{1}{r_1}\right)}{dn}=\frac{1}{r_1^2}\frac{R^2+r_1^2-l_1^2}{2Rr_1}.$$

Substituting these values and replacing r_1 and l_1 by their values $\frac{Rr}{l}$ and $\frac{R^2}{l}$ respectively, we have

(6) $$4\pi V(a,b,c)=\frac{1}{R}\iint\frac{R^2-l^2}{r^3}V d\sigma.$$

This is the form of the solution if there exists one. It, remains to prove that this solution satisfies the given conditions; *viz.*:

(i) V (a,b,c) is continuous and harmonic within the sphere,

(ii) that V (a,b,c) assumes at every point on the surface of the sphere the value that has been assigned to it at the point and that has been used in the calculation of the integral, *i.e.* to say, that, if $(a_0, b_0; c_0)$ be the co-ordinates of a point A_0 on the surface of the sphere and V_0 be the value assigned at A_0, V (a, b, c) tends to V_0 as the point (a,b,c) tends to (a_0, b_0, c_0) along any path.

5. Now, so long as (a,b,c) remains in the interior of the sphere r is not zero; therefore the integral considered as a function of (a, b, c) and its successive partial derivatives with respect to (a, b, c) are continuous in the interior of the sphere.

Also the integral satifies the equation of Laplace, *i.e.* to say

$$\frac{\partial^2 V}{\partial a^2}+\frac{\partial^2 V}{\partial b^2}+\frac{\partial^2 V}{\partial c^2}=0.$$

For, let f denote $\frac{R^2-l^2}{r^3}$, then, remembering that $R^2=x^2+y^2+z^2$;

$$l^2=a^2+b^2+c^2;\ r^2=(x-a)^2+(y-b)^2+(z-c)^2$$

where x, y, z are the co-ordinates of any point on the surface of the sphere

$$\frac{\partial^2 f}{\partial a^2}=-\frac{2}{r^3}+\frac{12a(a-x)}{r^5}+(R^2-l^2)\left\{-\frac{3}{r^5}+\frac{15}{r^7}(x-a)^2\right\};$$

similarly, we get $\frac{\partial^2 f}{\partial b^2}$ and $\frac{\partial^2 f}{\partial c^2}$; and we can easily see, if we note that

$$l^2+r^2-R^2=2[a(a-x)+b(b-y)+c(c-z)],$$

that

$$\frac{\partial^2 f}{\partial a^2}+\frac{\partial^2 f}{\partial b^2}+\frac{\partial^2 f}{\partial c^2}=0.$$

$$\therefore\quad \frac{\partial^2 V}{\partial a^2}+\frac{\partial^2 V}{\partial b^2}+\frac{\partial^2 V}{\partial c^2}=\frac{1}{4\pi R}\iint\left(\frac{\partial^2 f}{\partial a^2}+\frac{\partial^2 f}{\partial b^2}+\frac{\partial^2 f}{\partial c^2}\right)V d\sigma=0$$

i.e. $V(a, b, c)$ satisfies the equation of Laplace.

6. Lastly we have to show that

$$V(a, b, c),\ i.e.\ \frac{1}{4\pi R}\iint\frac{R^2-l^2}{r^3}V d\sigma$$

tends to V_o as (a, b, c) tends to (a_o, b_o, c_o)

Now $\frac{1}{4\pi R}\iint\frac{R^2-l^2}{r^3}d\sigma$ may be put equal to the sum of two integrals, viz.

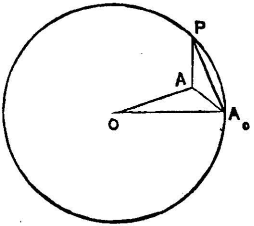
$$\frac{1}{4\pi R}\iint\frac{(R^2-l^2)}{r^3}V_o d\sigma+\frac{1}{4\pi R}\iint\frac{R^2-l^2}{r^3}(V-V_o)\ d\sigma.$$

The first integral represents a harmonic function continuous within the sphere and taking the constant value V_o on the sphere, assuming that such a function exists. It is evident that such a function exists since every constant satisfies the equation of Laplace. The first integral is thus equal to V_o. [This may also be shown by actual integration extended to the whole sphere, V_o being constant].

Therefore it remains only to prove that the second integral tends to zero.

Since the values assigned to V on the sphere vary in a continuous manner, their moduli must be less than a fixed quantity, say M. Also if we describe round the point A_o a small circle whose arcual radius subtends an angle δ at the centre, this circle divides the spherical surface into two parts; one is a small cap round A_o we shall denote it by c, and the other is the rest of the sphere, which we shall denote by C. Since V varies continuously on the sphere in the cap c, $(V-V_o)<\epsilon$ where ϵ is a small quantity which vanishes with δ,

In the rest of the spherical surface, *i.e.* in C, the distance of any point P on it from A_0 will be not less than $2\,R \sin \frac{1}{2}\,\delta$.; also if η be the distance of A from A_0, $r(=AP)$ will be not less than $A_0P - A_0A$, *i.e* not less than $2\,R \sin \frac{1}{2}\,\delta - \eta$. Lastly, $l(=OA)$ will be not less than $R-\eta$ and hence



$$R^2 - l^2 \leqq (2R-\eta)\,\eta.$$

Now going back to our second integral

$$\frac{1}{4\pi R}\iint \frac{R^2-l^2}{r^3}\,(V-V_0)\,d\sigma,$$

we may decompose it into two others, corresponding to the two regions c and C. Now the integral corresponding to small c, viz.

$$\frac{1}{4\pi R}\iint_c \frac{R^2-l^2}{r^3}\,(V-V_0)\,d\sigma$$

is in absolute value

$$<\frac{1}{4\pi R}\iint_c \frac{R^2-l^2}{r^3}\,\epsilon\, d\sigma.$$

i.e.

$$<\frac{\epsilon}{4\pi R}\iint_c \frac{R^2-l^2}{r^3}\,d\sigma$$

$$<\frac{\epsilon}{4\pi R}\iint_{c+C} \frac{R^2-l^2}{r^3}\,d\sigma$$

$<\epsilon$ [the integral extended to the sphere being $4\pi R$.]

Now the integral corresponding to C, viz.

$$\frac{1}{4\pi R}\iint_C \frac{R^2-l^2}{r^3}(V-V_o)\,d\sigma < \frac{1}{4\pi R}\iint_C \frac{(2R-\eta)\,\eta.\,2M.}{(2R\sin\frac{1}{2}\,\delta-\eta)^3}\,d\sigma$$

$$<\frac{2M(2R-\eta)\,\eta}{4\pi R(2R\sin\frac{1}{2}\,\delta-\eta)^3}\iint_C d\sigma$$

$$<\frac{2MR.(2R-\eta)\,\eta}{(2R\sin\frac{1}{2}\,\delta-\eta)^3}.$$

This last expression tends to zero if we decrease η sufficiently rapidly compared with δ, *i.e.* so rapidly that $\frac{\eta}{R}$ is of the order δ^4 at least.

Thus both the parts into which the second integral was broken up tend to zero. Therefore the integral itself tends to zero. Therefore it follows that $V(a,\,b,\,c)$ as given by the formula (6) considered as a

function of (a, b, c) is harmonic in the interior of the sphere and tends to the assigned value at any point of the surface as (a, b, c) tends to that point. Thus the interior problem of Dirichlet is completely solved in the case of a sphere.

P. V. SESHU AIYAR.

The Face of the Sky for October, November & December.

The Sun

enters Capricornus on December 23 at 3·46 A.M.

Phases of the Moon.

	October.				*November.*				*December.*			
	D.	H.	M.		D.	H.	M.		D.	H.	M.	
Last Quarter ...	1	9	44	A.M.								
New Moon ...	8	9	42	P.M.	7	7	52	P.M.	6	6	4	A.M.
First Quarter ...	15	1	52	„	13	11	3	A.M.	13	11	38	P.M.
Full Moon ...	23	0	16	A.M.	21	5	36	„	21	0	52	P.M.
Last Quarter ...	31	4	40	„	29	10	11	„	29	0	59	„

Planets.

Mercury is an evening star in early October. Towards the end of the month in November and in the early part of December, it is a morning star. It attains its greatest elogation (18° 57′ W) on November 7. It is in conjunction with the Moon on November 6 and December 6.

Venus is an evening star in these months. It is in conjunction with the Moon on October 9 at 8·0 P.M., on November 8 at 4·6 P.M., and on December 8 at 10·6 P.M. and with α Librae on October 27 at 1·30 A.M.

Mars is in Leo throughout these months. It is in conjunction with the Moon on October 3 at 7 11 A.M. on October 31, on November 9 at 5·47 A.M. and on December 27 at 0·38 A.M.; with Neptune on October 12 at 0·15 A.M.

Jupiter is in Pisces throughout these months. It is in retrograde motion in October and the first half of November. It is in conjunction with the Moon on October 20 at 7·55 A.M., on November 16 and on December 13 at 8·13 P.M.

Saturn is in Taurus throughout these months. It is in retrograde motion in November and December. It is in conjunction with the Moon on October 2 at 7·5 A.M. on October 29, on November 25 at 10·38 P.M. and on December 23 at 1·12 A.M.

Uranus is in conjunction with the Moon on October 17 and on December 13 at 11·13 P.M.

Neptune is in conjunction with the Moon on December 24.

V. RAMESAM.

SOLUTIONS.

Question 540.

(S. KRISHNASWAMI AIYANGAR) :—If S, H be the foci of the maximum inscribed ellipse of ABC and SH subtend angles $\theta_1, \theta_2, \theta_3$ at the vertices, prove that $bc\cos\theta_1 + ca\cos\theta_2 + ab\cos\theta_3 = \Sigma a^2 - \{\Sigma a^4 - \Sigma a^2 b^2\}^{\frac{1}{2}}$.

Solution by N. Sankara Aiyar, M. A.

We have $bc\cos\theta_1 = 3AS.AH\cos\theta_1$ (J .I.M.S., Vol. V, p. 14)

$$= \tfrac{3}{2}(AS^2 + AH^2 - SH^2)$$
$$= \tfrac{3}{2}(2AG^2 - 2GH^2)$$
$$= 3(AG^2 - GH^2)$$
$$= 3\left\{\frac{4}{9}\cdot\frac{2(b^2+c^2)-a^2}{4} - (\alpha^2 - \beta^2)\right\}$$
$$= \frac{2(b^2+c^2)-a^2}{3} - 3\{\alpha^1 + \beta^2)^2 - 4\alpha^2\beta^2\}^{\frac{1}{2}}.$$

$$\therefore\quad \Sigma bc\cos\theta_1 = \sum\frac{2(b^2+c^2)-a^2}{3} - 9\{(\alpha^2+\beta^2)^2 - 4\alpha^2\beta^2\}^{\frac{1}{2}}.$$
$$= (a^2+b^2+c^2) - \{(2\Delta\cot w)^2 - 12\Delta^2\}^{\frac{1}{2}}.$$
$$= (a^2+b^2+c^2) - \left\{\frac{(\Sigma a^2)^2}{4} - \frac{3}{4}16s(s-a)(s-b)(s-c)\right\}^{\frac{1}{2}}$$
$$= a^2+b^2+c^2 - \left\{\frac{\Sigma a^4 + 2\Sigma a^2b^2}{4} - \frac{3}{4}.(2\Sigma a^2b^2 - \Sigma a^4)\right\}^{\frac{1}{2}}$$
$$= a^2+b^2+c^2 - \{\Sigma a^4 - \Sigma a^2b^2\}^{\frac{1}{2}}.$$
$$= \Sigma a^2 - \{\Sigma a^4 - \Sigma a^2b^2\}^{\frac{1}{2}}.$$

Question 549.

(K. J. SANJANA, M.A.) :—If $\sum_n^r = 1^r - 2^r + 3^r + \ldots (-)^{n+1}n^r$, prove that

$$\sum_n^r + \binom{r+1}{1}\sum_{n-1}^r + \binom{r+2}{2}\sum_{n-2}^r + \ldots = 0,$$

if r is even and n is any integer equal to or greater than r. Show how the value of the series may be found when r is odd.

Remarks and Solution by K. B. Madhava.

It appears to me that this is not true for finite values of n. Actual working with cases where $r=2, 4, 6\ldots$ and n equal to a small number greater than the corresponding value of r say, 8, 6..., shows that this is not true at all, except in the trivial case when $r=2$ and n is any even number.

Any meaning can therefore be attached only when in the question n is supposed to tend to infinity and with that understanding Cesaro's method of summation of Asymptotic series § can be used to get the required result.

I shall slightly change the notation for convenience into the one which is more usual, *viz.* powers of $r-1$ instead of r, with corresponding changes throughout.

$$\underset{n\to 1}{\text{Lt.}} \sum u_n x^n \text{ is defined as } \underset{n\to\infty}{\text{Lt.}} \frac{S_n^{(r)}}{A_n^{(r)}}$$

where $\overset{(r)}{\underset{n}{S}} = S_n + \binom{r}{1} S_{n-1} + \binom{r+1}{2} S_{n-2} + \ldots + \binom{r+n-1}{2} S_o$ and

$$\overset{r}{\underset{n}{A}} = \frac{(r+1)(r+2)\ldots.(r+n)}{n\,!}$$

But $1-2^{r-1}x^2+3^{r-1}x^3-4^{r-1}x^4+\ldots$ is $r-$ply indeterminate and is of the form $\frac{P(x)}{(1+x)^r}$ where P is a polynomial of degree $r-2$.

Thus

$$(1-x)^{-r-1}[(1-2^{r-1}x+3^{r-1}x^2-\ldots)-l] = Q(x)(1-x^2)^{-r}$$

where Q is a polynomial of degree $r-1$ and

$$l = 1-2^{r-1}+3^{r-1}-\ldots = l$$

$$= \lim_{t\to 0} \left\{ e^t - 2^{r-1}e^{2t} + 3^{r-1}e^{3t} \ldots \right\}$$

$$= \lim_{t\to 0} \frac{d^{r-1}}{dt^{r-1}}\left(\frac{e^t}{1+e^t}\right).$$

Again expanding

$$\frac{e^t}{1+e^t} = \tfrac{1}{2} + B_1\,(2^2-1)\frac{t}{2!} - B_2(2^4-1)\frac{t^3}{4!} + B_3(2^6-1)\frac{t^5}{6!}\ldots$$

so that $\lim_{t\to 0} \frac{d^{r-1}}{dt^{r-1}}\left(\frac{e^t}{1+e^t}\right) = \left(2^{2n}-1\right)\frac{B_n}{2n}$ if $r=2n$;

and if r is odd, $\lim \overset{r}{\underset{n}{S}} = 0$ as was required to be established.

Hence if r is even and equal to $2n$, $\lim_{n\to 0} \overset{(r)}{\underset{n}{A}} = \frac{1}{r!}$, so that

$$\lim \overset{r}{\underset{n}{S}} = r!\; l = (2n-1)!\,(2^{2n}-1)\,B_n.$$

(§) Bromwich §§ (122—8)

Question 567.

(P. A. Subramanya Aiyar, B.A., L.T.) :—Show that a uniform tetrahedron of mass m is equi-momental with six particles of mass $\frac{1}{10} m$ at the middle points of its six edges and one of mass $\frac{2}{5} m$ at the centre of mass of the tetrahedron.

Solution by K. B. Madhava and D. Krishnamurti.

First, it is obvious that the two systems have the same centre of inertia.

Let OABC be the tetrahedron; P, Q, R the midpoints of OA, OB, OC; D, E, F of BC, CA, AB. Replace the particles (each of mass $\frac{1}{10} m$) at D, E and F by the equivalent particles at vertices A, B and C *viz*, by particles each of mass $\frac{1}{20} m$. Also replace those at P, Q, R in the same way. Thus we have $\frac{3}{20} m$ at O and the same amount at each of the other three vertices. The latter are again together equivalent to $\frac{9}{20} m$ at g, the centre of gravity of the triangle ABC. Obviously then the centre of gravity of the whole system is at G, the *c.g.* of the tetrahedron itself.

If we now prove that the two systems have the same moments and products of inertia about some three axes, we can show that the two systems are kinetically equivalent.

Draw through O, three straight lines OX, OY, OZ meet at right angles choosing them as co-ordinate axes. Let (x_1), $(x_2)_s$ $x_3)$ be the co-ordinates of A, B, C respectively. Then $\left(\frac{x_1}{2}\right)$; $\left(\frac{x_2}{2}\right)$; $\left(\frac{x_3}{2}\right)$ are the co-ordinates of P, Q, R and $\left(\frac{x_2+x_3}{2}\right)$, $\left(\frac{x_3+x_1}{2}\right)$, $\left(\frac{x_1+x_2}{2}\right)$ of D, E, F.

Let p be the length of the $\perp^r$ from O on the opposite face and take a section LMN parallel to ABC at distance ξ from O. Its area is $\frac{A\xi^2}{p^2}$ where A is the area of $\triangle$ ABC. Now the moment of inertia of a thin slice of thickness $d\xi$ about OX

$=$ that of particles each $\frac{1}{3} A\rho \frac{\xi^2}{p^2} d\xi$ placed at the mid-points

$\triangle$ LMN.

$$=\frac{1}{3}A\rho\frac{\xi^2}{p^2}d\xi\left[\left(\frac{\xi}{p}\cdot\frac{y_2+y_3}{2}\right)^2+\left(\frac{\xi}{p}\cdot\frac{y_3+y_1}{2}\right)^2+\left(\frac{\xi}{p}\cdot\frac{y_1+y_2}{2}\right)^2\right.$$
$$\left.+\left(\frac{\xi}{p}\cdot\frac{z_2+z_3}{2}\right)^2+\left(\frac{\xi}{p}\cdot\frac{z_3+z_1}{2}\right)^2+\left(\frac{\xi}{p}\cdot\frac{z_1+z_2}{2}\right)^2\right]$$

Hence integrating *w. r. t.* ξ its moment of inertia about OX

$$=\frac{1}{5}A\rho\frac{p}{4}[\{(y_2+y_3)^2+(y_3+y_1)^2+(y_1+y_2)^2\}$$
$$+\{(z_2+z_3)^2+(z_3+z_1)^2+(z_1+z_2)^2\}].$$

$$=\frac{m}{10}[y_1^2+y_2^2+y_3^2+z_1^2+z_2^2+z_3^2+y_2y_3+y_3y_1+y_1y_2+z_2z_3+z_3z_1+z_1z_2].$$

Now, the moment of inertia about OX of six particles each of mass $\frac{m}{10}$ at P, Q, R, D, E, F and one of $\frac{2}{5}m$ at G

$$=\frac{m}{10}\left[\left\{\frac{y_1^2+z_1^2}{4}+\frac{y_2^2+z_2^2}{4}+\frac{y_3^2+z_3^2}{4}+\frac{(y_2+y_3)^2}{4}+\right.\right.$$
$$\left.\frac{(y_3+y_1)^2}{4}+\frac{(y_1+y_2)^2}{4}+\frac{(z_2+z_3)^2}{4}+\frac{(z_3+z_1)^2}{4}+\frac{(z_1+z_2)^2}{4}\right]$$
$$+\frac{2m}{5}\left\{\left(\frac{y_1+y_2+y_3}{4}\right)^2+\left(\frac{z_1+z_2+z_3}{4}\right)^2\right\}$$

$$=\frac{m}{10}[y_1^2+y_2^2+y_3^2+z_1^2+z_2^2+z_3^2+y_1y_2+y_2y_3+y_3y_1+z_1z_2+z_2z_3+z_3z_1]$$

which is the same as before.

Similarly we can show the equivalence of the moments of inertia about OY and OZ of the two systems.

In a similar manner the product of inertia of the tetrahedron about OY and OZ

$$=\frac{m}{20}[(y_2+y_3)(z_2+z_3)+(y_3+y_1)(z_3+z_1)+(y_1+y_2)(z_1+z_2)];$$

that of the seven particles

$$=\frac{m}{10}\left[\frac{y_2z_2}{4}+\frac{y_3z_3}{4}+\frac{y_1z_1}{4}+\frac{(y_2+y_3)(z_2+z_3)}{4}+\frac{(y_3+y_1)(z_3+z_1)}{4}\right.$$
$$\left.+\frac{(y_1+y_2)(z_1+z_2)}{4}\right]+\frac{2m}{5}\left[\left\{\frac{y_1+y_2+y_3}{4}\right\}\left\{\frac{z_1+z_2+z_3}{4}\right\}\right]$$

$$=\frac{m}{20}[(y_2+y_3)(z_2+z_3)+(z_3+z_1)(y_3+y_1)+(y_1+y_2)(z_1+z_2)],$$

which are equivalent.

Similarly, we can show the equivalence of the other two products of the two systems.

Hence the two systems are equi-momental [Routh: *Rigid Dynamics*, Vol. I. §§ 13, 15. Also Loney: *Dynamics of a Particle*, p. 198, Q. 12.]

Question 603.

(S. KRISHNASWAMI AIYANGAR) :—Show that

$$1-(x)^2+\left(\frac{x^2}{2\,!}\right)^2-\left(\frac{x^3}{3\,!}\right)^2\ldots=\frac{2}{\pi}\int_0^\pi \cos^2(x\sin\theta)d\theta-1,$$

and find the value of $1+x^2+\left(\frac{x^2}{2\,!}\right)^2+\left(\frac{x^3}{3\,!}\right)^2+\ldots\ldots$

Solution by P. V. Seshu Aiyar, K. J. Sanjana, T. P. Trivedi and N. Sankara Aiyar.

The right hand side $=\frac{2}{\pi}\int_0^\pi \frac{1+\cos 2(x\sin\theta)}{2}d\theta-1$

$$=\frac{2}{\pi}\int_0^\pi \left\{\frac{d\theta}{2}+\frac{\cos(2x\sin\theta)}{2}d\theta\right\}-1$$

$$=\frac{1}{\pi}\int_0^\pi \cos(2x\sin\theta)d\theta. \qquad (1)$$

Now $\cos(2x\sin\theta)=1-\frac{(2x)^2}{2\,!}\sin^2\theta+\frac{(2x)^4}{4\,!}\sin^4\theta-\ \ldots\ \ \ldots$

Hence the general term on the R. H. S. of (1) is

$$\frac{1}{\pi}\int_0^\pi (-1)^n\frac{(2x)^{2n}}{(2n)\,!}\sin^{2n}\theta d\theta$$

$$=(-1)^n\frac{(2x)^{2n}}{(2n)\,!}\,\frac{1}{\pi}\int_0^\pi \sin^{2n}\theta d\theta$$

$$=(-1)^n\frac{(2x)^{2n}}{(2n)\,!}\,\frac{1}{\pi}\cdot\frac{(2n-1)(2n-3)\ldots\ldots1}{2\,n(2n-2)\ldots\ldots\ldots\ldots2}\,\pi.$$

$$=(-1)^n\frac{2^{2n}x^{2n}}{2n(2n-2)\ldots2}\cdot\frac{1}{2n(2n-2)\ldots2}$$

$$=(-1)^n\frac{x^{2n}}{\{n(n-1)\ldots1)\}^2}=(-1)^n\left(\frac{x^n}{n\,!}\right)^2.$$

which is the general term on the left. Hence the given result.

By putting ix in the place of x we get

$$1+(x)^2+\left(\frac{x^2}{2\,!}\right)^2+\ldots=\frac{2}{\pi}\int_0^\pi \cosh(2x\sin\theta)d\theta.$$

Note.—The L. H. S. in the question is only $J_0\,(2x)$ and the R. H. S is the well-known integral form for J_0

Remarks by K. B. Madhava.

The first expression is the Bessel Function $J_o(2x)$ and is therefore equal to the integral on the right hand side which is more familiarly written

$$\frac{1}{\pi}\int_{o}^{\pi} \cos(2x \sin\theta)\, d\theta.$$

The second expression is $J_o(2ix)$, for some interesting remarks on which reference may be made to Byerly's "*Fourier's Series and Spherical Harmonics*", page 232.

Question 605.

(S. Ramanujan) :—Shew that, when $x=\infty$

$$\frac{(x+a-b)!\ (8x+2b)!\ (9x+a+b)!}{(3x+a-c!\ (3x+a-b+c)!\ (12x+3b)!}=\sqrt{\frac{2}{3}}.$$

Solution by K. J. Sanjana and A. Narasinga Rao.

By Stirling's Theorem, the limit $(x\rightarrow\infty)$ of $(x+a-b)!$ is

$$\sqrt{\{2\pi(x+a-b)\}}.\ \left\{\frac{x+a-b}{e}\right\}^{x+a-b};$$

and similar results hold for the other factorials. Cancelling $(2\pi)^{\frac{3}{2}}$ it is seen that the limit required is the product of the limits of

$$\sqrt{\frac{(x+a-b)(8x+2b)(9x+a+b)}{(3x+a-c)(3x+a-b+c)(12x+3b)}}$$

and

$$\frac{(x+a-b)^{x+a-b}(.\ ..\ \ .)(.\ \ .\ \ .)}{(3x+a-c)^{3x+a-c}(.\ \ .\ \ .)(.\ \ .\ \ .)}.$$

The first of these $=\sqrt{\dfrac{72+\frac{a_1}{x}+\frac{b_1}{x^2}+\frac{c_1}{x^3}}{108+\frac{a_2}{x}+\frac{b_2}{x^2}+\frac{c_2}{x^3}}}=\sqrt{\dfrac{2}{3}}.$

For the second, we see that

$$(x+a-b)^{x+a-b}=x^{x+a-b}\left\{1+\frac{a-b}{x}\right\}^{x+a-b}$$

$$=x^{x+a-b}\left\{1+\frac{a-b}{x}\right\}^{\frac{x}{a-b}\left\{a-b+\frac{(a-b)^2}{x}\right\}}$$

$$=x^{x+a-b}.e^{a-b+\frac{(a-b)^2}{x}};$$

there will be similar expressions for the other factors.

Thus the second limit

$$=\frac{x^{18x+2a+2b}.8^{8x+2b}.9^{2x+a+b}.e^{2a+2b}}{x^{18x+2a+2b}.3^{x+a-c}.3^{8x+a-b+c}.12^{12x+3b}.e^{2a+2b}}$$

$$=\frac{2^{24x+6b}.3^{18x+2a+2b}}{3^{18x+2a+2b}.2^{24x+6b}}=1.$$

Hence the given result is obtained.

Question 606.

(S. Ramanujan) :—Shew that

$$\sum_{0}^{\infty}\left\{\frac{(\sqrt{5}-2)^{2n+1}}{(2n+1)^2}\right\}=\frac{\pi^2}{24}-\frac{1}{12}\left\{\log(2+\sqrt{5})\right\}^2.$$

Solution by N. Sankara Aiyar, M.A.

Now

$$\sum_{0}^{\infty}\frac{(\sqrt{5}-2)^{2n+1}}{(2n+1)^2}=f(\sqrt{5}-2)$$

$$=\frac{\pi^2}{24}-\frac{3}{4}\left\{\log\frac{\sqrt{5}-1}{2}\right\}^2. \text{ (J. I. M. S., Vol. VII, p. 34)}$$

$$=\frac{\pi^2}{24}-\frac{1}{12}\left\{\log\left(\frac{\sqrt{5}-1}{2}\right)^3\right\}^2.$$

$$=\frac{\pi^2}{24}-\frac{1}{12}\left\{\log\frac{5\sqrt{5}-3\cdot5+3\sqrt{5}-1}{8}\right\}^2$$

$$=\frac{\pi^2}{24}-\frac{1}{12}\left\{\log(\sqrt{5}-2)\right\}^2$$

$$=\frac{\pi^2}{24}-\frac{1}{12}\left\{\log(\sqrt{5}+2)\right\}^2$$

Question 610.

(K. V. Anantanarayana Sastri, B.A.) :—If T is a point on the directrix of a parabola and t_1, t_2, the lengths of the tangents TP, TQ from it to the parabola, shew that the length of the arc PQ of the parabola is

$$\frac{t_1^3+t_2^3}{t_1^2+t_2^2}+\frac{t_1^2t_2^2}{(t_1^2+t_2^2)^{3/2}}\left\{\sinh^{-1}\frac{t_1}{t_2}+\sinh^{-1}\frac{t_2}{t_1}\right\}.$$

Solution by K. J. Sanjana, M.A. and S. Venkataraya Sastry, M.A., L.T.

With TP, TQ as axes, the equation of the curve is

$\left(\frac{x}{t_1}\right)^{\frac{1}{2}}+\left(\frac{y}{t_2}\right)^{\frac{1}{2}}=1$; and the required length of arc is

$$\int_0^{t_1}\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}}dx,$$

Putting $x=t_1z^2, y=t_2(1-z)^2$, $-z$ being a positive proper fraction, we get the integral

$$= \int_0^1 \{ (t_1^2+t_2^2)z^2 - 2t_2^2z + t_2^2 \}^{\frac{1}{2}} dz,$$

It will be found that the indefinite integral is

$$\phi(z) \equiv \tfrac{1}{2}\left(z - \frac{t_2^2}{t_1^2+t_2^2}\right)\sqrt{\{ (t_1^2+t_2^2)z^2 - 2t_2^2z + t_2^2 \}}$$
$$+ \tfrac{1}{2}\sqrt{(t_1^2+t_2^2)}\frac{t_1^2t_2^2}{(t_1^2+t_2^2)^2} \sinh^{-1}\frac{(t_1^2+t_2^2)z - t_2}{t_1t_2},$$

omitting a constant.

Thus the length of arc is

$2\phi(1) - 2\phi(0)$

$$= \frac{t_1^3}{t_1^2+t_2^2} + \frac{t_1^2t_2^2}{(t_1^2+t_2^2)^{3/2}} \sinh^{-1}\frac{t_1}{t_2} - \left\{ \frac{-t_2^3}{t_1^2+t_2^2} + \frac{t_1^2t_2^2}{(t_1^2+t_2^2)^{3/2}} \sinh^{-1}\frac{-t_2}{t_1} \right\}$$
$$= \frac{t_1^3+t_2^3}{t_1^2+t_2^2} + \frac{t_1^2t_2^2}{(t_1^2+t_2^2)^{3/2}} \left\{ \sinh^{-1}\frac{t_1}{t_2} + \sinh^{-1}\frac{t_2}{t_1} \right\}.$$

Additional Solution by L. N. Subramanian, M.A., L.T.

Question 612.

(N. Ganapathi Subba Aiyar) :—A series of ellipses have a common focus, pass through a given point and have their major axes of the same length. Find their envelope.

Solution (1) *by A. Narasinga Rao, N. Durai Rajan and* **N.** *B. Pendse*; (2) *by K. J. Sanjana, M.A., T. P. Trivedi, M.A., LL.B.*

(1) Let F be the common focus, P the given point and $2a$ the length of the major axis. If H is the second focus of one of the ellipses thus $FP+PH=2a$. The locus of H is a circle, centre P, and radius $r=2a-FP$.

Let H, H′ be consecutive points on this circle and Q a common point of the ellipses corresponding to H and H′, so that the locus of Q is the required envelope.

We have $FQ+QH=2a=FQ+QH'$. $\therefore$ $QH=QH'$.

Hence QH is ultimately normal to the circle and passes through P. Also $PQ+QF=r+2a=4a-FP$.

The required envelope is therefore an ellipse with F and P for foci, whose major axis is $4a-FP$.

This result may be generalised thus :—

The bipolar equation of a curve is $r'=f(r)$. If one of the poles F be fixed, the envelope of the curves which pass through a given point

P is a set of curves whose equation referred to F and P as poles is of the form $r'=f(r)+\text{const.}$

With a notation similar to the above, we have $PH=f(PF)$, and the locus of H is a circle or a series of concentric circles according as f is single or multiple-valued.

Also $\qquad PH=f(PF),\ QH=f(QF).$

$\therefore \qquad PQ=f(QF)+\text{const.}$

Hence the locus of Q is a series of curves of the form

$$r'=f(r)+\text{const.}$$

(2) Take the common focus S as origin and SP as the initial line, where P is the given common point of the ellipses.

The equation of any conic is

$$\frac{a(1-e^2)}{r}=1+e\cos(\theta-\alpha),$$

where a is the constant semi-major axis, e the eccentricity, and α the angle made by the axis with SP.

Let $SP=c$; since P lies on the conic

$$\frac{a(1-e^2)}{c}=1+e\cos\alpha$$

or

$$\cos\alpha=\frac{a(1-e^2)-c}{ce}.$$

Substituting in the equation of the conic, transposing and squaring,

$$\left\{\frac{a(1-e^2)}{r}-1-\frac{a(1-e^2)-c}{c}\cos\theta\right\}^2$$

$$=e^2\sin^2\theta\sin^2\alpha=e^2\sin^2\theta\,(1-\cos^2\alpha)$$

$$=e^2\sin^2\theta-\left\{\frac{a(1-e^2)-c}{c}\right\}^2\sin^2\theta,$$

whence

$$(1-e^2)^2\{a^2(c^2-2cr\cos\theta+r^2)\}-(1-e^2)\{2acr(c+r)(1-\cos\theta)\}$$
$$-r^2c^2(1-\cos\theta)(1+\cos\theta)+r^2c^2(1-\cos\theta)^2=0.$$

Taking the discriminant of $(1-e^2)$, we get for the envelope

$$\{2a(c+r)-rc(1+\cos\theta)\}^2=4a^2(c^2-2cr\cos\theta+r^2),$$

or

$$8a^2+rc(1+\cos\theta)-4a(c+r)=0,$$

or

$$(8a^2-4ac+cx)^2=(4a-c)^2(x^2+y^2),$$

which is a conic section.

Question 615.

(Selected):—Find two numbers such that the sum of all the aliquot parts of either is equal to the other.

Solution by K. Ramaswami Iyengar, B.A., B.L.

Suppose the numbers required to be $N=2^n.p.q$ and $N'=2^n.r$ where p, q, r are primes.

Since the sum of the aliquot parts of one is equal to the other, we have the following equations involving p, q, r.

$$(1+2+2^2+\ldots 2^n)(1+p)(1+q)-\mathrm{N}=\mathrm{N}',$$
$$(1+2+\ldots 2^n)(1+r)-\mathrm{N}'=\mathrm{N}.$$
$$\therefore \quad (2^{n+1}-1)(1+p)(1+q)=\mathrm{N}+\mathrm{N}'$$
$$=2^n(pq+r)=(2^{n+1}-1)(1+r).$$

Hence we get, $(p+1)(q+1)=(r+1)$ ... ... (1)

$$\frac{1}{p+1}+\frac{1}{q+1}=\frac{1}{2^n} \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

Now, to satisfy (2) put $p+1=2^n(2^k+1)$,
$q+1=2^n(2^{-k}+1)$.

Taking a very simple case, when $k=0$,

$$p+1=3\cdot 2^n \text{ and } q+1=3\cdot 2^{n-1}.$$

The numbers are

$$\mathrm{N}=2^n(3\cdot 2^n-1)(3\cdot 2^{n-1}-1)$$

and
$$\mathrm{N}'=2^n(3^2\cdot 2^{2n-1}-1)$$

where n is so chosen as to make $3\cdot 2^n-1$, $3\cdot 2^{n-1}-1$, $3^2\cdot 2^{2n-1}-1$ primes.

When $n=2$, 4 the numbers are (220, 284) and (17296, 18416).

Question 616.

(S. Narayana Aiyar, M. A.) :—Establish the following extension of Maclaurin's theorem

$$\phi(x)+\frac{a-b}{b}\cdot\frac{x}{1!}\phi'(x)+\frac{(a-b)(a-b-1)}{b(b+1)}\cdot\frac{x^2}{2!}\phi''(x)+\ldots$$
$$=\phi(0)+\frac{a}{b}\cdot\frac{x}{1!}\phi'(0)+\frac{a(a+1)}{b(b+1)}\cdot\frac{x^2}{2!}\phi''(0)+\ldots$$

(*a*) When $a=b$, this reduces to Maclaurin's theorem.

(*b*) When $\phi(x)=e^x$, this reduces to Question 576 (J. I. M. S.)

(*c*) Examine the case when $\phi(x)=\cos x$, $(1+x)^{-n}$ *etc.*

Solution by K. B. Madhava, E. J. Pocock, T. P. Trivedi, and M. Bhimasena Rao.

This is obtained by a repeated application of Maclaurin's theorem.

What we have to prove is merely

$$\frac{1}{n!}+\frac{a-b}{b}\cdot\frac{1}{1!}+\frac{(a-b)(a-b-1)}{b(b+1)}\cdot\frac{1}{2!}+\frac{(a-b)(a-b-1)(a-b-2)}{b(b+1)(b+2)}\cdot\frac{1}{3!}+\ldots$$
$$=\frac{a(a+1)(a+2)\ldots(a+n-1)}{b(b+1)(b+2)\ldots(b+n-1)}\cdot\frac{1}{n!};$$

and this is but a slight alteration of Mr. Sanjana's problem No. 16755, solved in the "Educational Times" Vol. 18, p. 93, with his b and c respectively $-(a-b)$ and b of this problem.

(a) Obviously it is Maclaurin's theorem when $a=b$; for, all the terms, except the first, on the L.H.S. vanish.

(b) As the proposer himself points out, if $\phi(x)=e^x$ it is his Question 576 for e^x is not altered in its form by any number of differentiations.

(c) With $\phi(x)=\cos x$, we get

$$\cos x-\frac{(a-b)}{b}\,\frac{x}{1\,!}\sin x-\frac{(a-b)(a-b-1)}{b(b+1)}\cdot\frac{x^2}{2\,!}\cos x+\ldots$$
$$=1.\frac{a(a+1)}{b(b+1)}\frac{x^2}{2\,!}+\frac{a(a+1)(a+2)(a+3)}{b(b+1)(b+2)(b+3)}\cdot\frac{x^4}{4\,!}-\ldots$$

as we should expect it without odd powers on the R.H.S.; it is but a most general example which would come under his formula (3) which he has obtained on p. 184 of the J. I. M. S. for October 1913.

(d) With $\phi(x)=(1+x)^{-n}$, we get

$$\frac{1}{(1+x)^n}\left\{1-\frac{a-b}{b}\cdot\frac{n}{1}\cdot\frac{x}{1+x}+\frac{(a-b)(a-b-1)n(n+1)}{b(b+1)\quad 1\cdot 2}\left(\frac{x}{1+x}\right)^2\ldots\right\}$$
$$=1-\frac{a}{b}\cdot\frac{n}{1}\cdot x+\frac{a(a+1)}{b(b+1)}\cdot\frac{n(n+1)}{1\cdot 2}x^2-\ldots$$

which, reducing it to Mr. Sanjana's form, is

$$\frac{1}{(1+x)^n}\left\{1+\frac{b}{c}\cdot\frac{n}{1}\left(\frac{x}{1+x}\right)+\frac{b(b+1)}{c(c+1)}\cdot\frac{n(n+1)}{1\cdot 2}\cdot\left(\frac{x}{1+x}\right)^2\ldots\right\}$$
$$=1-\frac{(c-b)}{c}\cdot\frac{n}{1}\cdot+\frac{(c-b)(c-b+1)}{c(c+1)}\cdot\frac{n(n+1)}{1\cdot 2}\cdot x^2-\ldots$$

which states that

$$F(n,\ c-b,\ c,-x)=\frac{1}{(1+x)^n}F\left(n,\ b,\ c,\frac{x}{1+x}\right)$$

with the usual notation F for the hypergeometric series.

This result viz:

$$F(\alpha,\ \beta,\ \gamma,\ x)=\frac{1}{(1-x)^{\alpha}}F\left(\alpha,\ \gamma-\beta,\ \gamma,\frac{x}{x-1}\right)$$

can be more easily established by Euler's Transformation as suggested by Ex. 9. p. 170 of Bromwich.

QUESTIONS FOR SOLUTION.

686. (S. NARAYANA AIYAR, M. A.):—Establish the following identities:—

i. $$\frac{a(a+1)(a+2)\dots(a+n-1)}{b(b+1)(b+2)\dots(b+n-1)}=1+{}_nC_1.\frac{a-b}{b}+{}_nC_2.\frac{(a-b)(a-b-1)}{b(b+1)}+\dots\dots+\frac{(a-b)(a-b-1)\dots(a-b-n+1)}{b(b+1)\dots\dots(b+n-1)}.$$

ii. $$\frac{(a-b)(a-b-1)(a-b-2)\dots(a-b-n+1)}{b(b+1)(b+2)\dots(b+n+1)}=(-1)^n.\left\{1-{}_nC_1.\frac{a}{b}+nC_2.\frac{a(a+1)}{b(b+1)}-\dots(-1)^n\frac{a(a+1)\dots(a+n-1)}{b(b+1)\dots(b+n-1)}\right\}.$$

iii. $F(a, b, c, x)=(1-x)^{-a}.\ F\left(a, c-b, c, \frac{x}{x-1}\right)$, where F stands for the hypergeometric series.

iv. $$\left\{x.\frac{a}{b}-\frac{x^2}{2}.\frac{a(a+1)}{b(b+1)}+\frac{x^3}{3}.\frac{a(a+1)(a+2)}{b(b+1)(b+2)}-\dots\dots\right\}-\left\{\frac{x}{1+x}.\frac{a-b}{b}-\frac{x^2}{2(1+x)^2}.\frac{(a-b)(a-b-1)}{b(b+1)}+\dots\dots\right\}=\log(1+x).$$

v. $$\pi.\frac{a}{b}-\frac{\pi^3}{3!}.\frac{a(a+1)(a+2)}{b(b+1)(b+2)}+\frac{\pi^5}{5!}.\frac{a(a+1)(a+2)(a+3)(a+4)}{b(b+1)(b+2)(b+3)(b+4)}-\dots\dots=-\left\{\pi.\frac{a-b}{b}-\frac{\pi^3}{3!}.\frac{(a-b)(a-b-1)(a-b-2)}{b(b+1)(b+2)}+\dots\dots\right\}$$

vi. $$\frac{\pi^2}{2!}.\frac{a(a+1)}{b(b+1)}+\frac{\pi^4}{4!}.\frac{a(a+1)(a+2)(a+3)}{b(b+1)(b+2)(b+3)}-\dots\dots=-\left\{1-\frac{\pi^2}{2!}.\frac{(a-b)(a-b-1)}{b(b+1)}+\dots\dots\right\}$$

687. (K. J. SANJANA, M.A.):—Find the envelope of the straight line $ul^2+vm^2+w+2u'm+2v'l+2w'lm=0$, where u, v, w, u', v', w' are given linear functions of the co-ordinates (x, y) and l, m are parameters connected by the relation $pl^2+qm^2=1$.

What will be the degree of the envelope equation when u, v, w, u', v', w' are all or some of them quadratic functions?

688. (GANPATRAM R. JANI):—S_r being the sum of the rth powers of the first n natural members, prove that

$$S_{r+1}=(r+1)\int^n S_r\,dn+qn.$$

When r is even, $q=0$; when r is odd, q may be found by giving numerical values to n.

689. (J. C. SWAMINARAYAN, M.A). :—If $S_m = 1^m + 2^m + 3^m + \dots + n^m$, prove that

$$7S_6 + 5S_4 = 12S_2S_1^2.$$

690. (J. C. SWAMINARAYAN, M.A.) :—If S_r denotes the sum of the reciprocals of the first r odd natural numbers, prove the following relations :—

(i) $\frac{S_1}{1\cdot 2} - \frac{S_2}{3\cdot 4} + \frac{S_3}{5\cdot 6} - \frac{S_4}{7\cdot 8} + \dots\dots\dots$

$$= -\frac{\pi^2}{32} + \frac{\pi}{8}\log 2 + \tfrac{1}{2}\left\{1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \dots\right\}$$

(ii) $\frac{1}{1\cdot 2} - \frac{S_1}{3\cdot 4} + \frac{S_2}{5\cdot 6} - \frac{S_3}{7\cdot 8} + \dots\dots\dots$

$$= -\frac{\pi^2}{32} + \frac{\pi}{4} + \left(\frac{\pi}{8} - \frac{1}{2}\right)\log_e 2 + \tfrac{1}{2}\left\{\frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{7^2} - \frac{1}{9^2} + \dots\right\}.$$

691. (A. NARASINGA RAO) :—A marble slab of n pounds breaks into k pieces with which a tradesman finds himself able to weigh goods from 1 to n pounds (fractions excluded). Show that the least value of k is the smallest integer satisfying the relation $3^k \geq 2n+1$.

What are the weights of the several pieces ?

692. (R. VYTHYNATHASWAMY, B.A.) :—Prove that there cannot exist a hyperboloid, which is such that three and therefore all generators of each system are nul lines *w. r. t.* a system of forces.

A system of forces acts along generators of one system of a hyperboloid and l, m are the two generators of that system which are nul lines. Shew that the nul plane of any point O is the plane OPQ where P, Q are the points of contact of the planes (l, O) (m, O). Prove also that, conversely, the nul point of any plane which cuts l, m in P, Q is the pole of PQ *w. r. t* the section.

693. (R. VYTHYNATHASWAMY, B.A.) :—If ABC, A′B′C′ be triangles inscribed in the same circle, L_1, L_2, L_3 the latera-recta of parabolas having A,B,C for foci and touching the sides of A′B′C′, L_1', L_2', L_3' the latera-recta of parabolas having A′,B′,C′, as foci and touching the sides of ABC, shew that $L_1L_2L_3 = L_1'L_2'L_3'$.

694. (M. BHIMASENA, RAO) :—Tangents at P and Q on a cubic curve intersect on the curve. Through Q a variable line cuts the curve in R and R′. Prove that PR and PR′ are conjugate lines of a pencil in involution,

Deduce that the four lines formed by joining any fixed point on a cubic curve to the points of contact of tangents drawn from a variable point on the curve (to itself), taken two by two, form three pencils in involution, and determine the double rays of the pencils.

695. (R. SRINIVASAN, M.A.) :—From a point P on a sphere four straight lines PA, PB, PC, PD are drawn to meet it in A, B, C, D such that $A\hat{P}B = B\hat{P}C = C\hat{P}D = D\hat{P}A$; shew that PA+PC=PB+PD.

696. (S. KRISHNASWAMY AIYANGAR) :—If λ, μ be the latera-recta of the parabola and the rectangular hyperbola of closest contact with a curve at any point, prove that $2\lambda\rho=\mu^2$.

697. (S. KRISHNASWAMY AIYANGAR) :—The base BC of a triangle is given in magnitude and position. If the intercept on the tangent to the incircle parallel to the base and terminated by the sides is of constant length, find the locus of the vertex.

698. (D. KRISHNAMURTI) :—A variable tetrahedron has its vertices lying on four fixed straight lines passing through a point O. If ABCD be the vertices in any position and $\alpha, \beta, \gamma, \delta$ the tetrahedral co-ordinates of a fixed point with reference to ABCD, show that

$$\frac{(OA \cdot OB \cdot OC \cdot OD)\alpha\beta\gamma\delta}{(\alpha \cdot OA + \beta \cdot OB + \gamma \cdot OC + \delta \cdot OD)^4} \text{ is constant.}$$

699. (S. RAMANUJAN) :—Show that the roots of the equations

(i) $x^6 - x^3 + x^2 + 2x - 1 = 0$

(ii) $x^6 + x^5 - x^3 - x^2 - x + 1 = 0$

can be expressed in terms of radicals.

700. (S. RAMANUJAN) :—Sum the series

$$(a+b+1)\left(\frac{a}{b}\right)^2 + (a+b+3)\left\{\frac{a(a+1)}{b(b+1)}\right\}^2 + (a+b+5)\left\{\frac{a(a+1)(a+2)}{b(b+1)(b+2)}\right\}^2 + \ldots \text{to } n \text{ terms.}$$

701. (K. APPUKUTTAN ERADY, M.A.) :—P is a point within the area bounded by two confocal ellipses of semi-axes (a, b) and (a', b'). Prove that, if S, H be the common foci, the mean value of $(SP.\ HP)^{-1}$ is

$$\frac{2}{a'b' - ab} \cdot \log \frac{(a'+b')}{(a+b)};$$

and that of (SP.HP) is

$$\frac{1}{4(a'b'-ab)}\left\{2(a^2-b^2)\log\left(\frac{a'+b'}{a+b}\right)+a'b'(a'^2+b'^2)-ab(a^2+b^2)\right\}$$

702. (Martyn M. Thomas) :—Prove that

(i) $\int_0^{\frac{\pi}{2}} \sin x\,(\log \sin x).\cos x \log(\cos x)\,dx = \frac{12-\pi^2}{48}$.

(ii) $\int_0^{\infty} \frac{\sin x \log(\cos x)}{x}\,dx = \pi \log \frac{1}{\sqrt{2}}$.

703. (N. Sankara Aiyar, M.A.) :—If AP is the symmedian through A, prove that

$$\Sigma\{(b^2+c^2)\text{AK.KP}\} = 6\,a^2b^2c^2.$$

704. (S. Malhari Rao, B.A.) :—If the sum of a number of three digits and the number formed by reversing the digits be divisible by 37, shew that the sum of all such pairs of number is 480×37.

705. (S. Malhari Rao, B.A.) :—The circumferences of four concentric circles are cut by two diameters. Arrange the numbers 1, 2, 3,...16 at the points of intersection in such a way that the sum of the numbers on each radius and on each circumference may be the same, and also the sum of every pair of diametrically opposite numbers may be the same.